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Three-component nonlinear dynamical system generated by the new third-order discrete spectral problem

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Abstract

We propose a nonlinear model on a regular infinite one-dimensional lattice. It describes the three-component dynamical system with modulated on-site masses and is shown to admit a zero-curvature representation. The associated auxiliary spectral problem is basically of third order and gives rise to fairly complicated subdivision into domains of regularity of Jost functions in the plane of complex spectral parameter. As a result, both the direct and the inverse scattering problems turn out to be substantially nontrivial. The Caudrey version of the direct and inverse scattering technique for the needs of model integration is adapted. The simplest soliton solution is found.

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1. Introduction

In starting to compose the material for this paper, we were fully aware of main driving forces enabling us to formulate a more or less coherent idea for this work and the basic tools of its implementation. Since 1982, when becoming acquainted with the two-component Davydov–Kyslukha model [1–3] invoked to explain a possible mechanism of energy and charge transfer in long regular macromolecules, and especially since 1984 when becoming acquainted with the integrable discretization of the nonlinear Schrödinger equation given by Ablowitz and Ladik [4–6], the inexplicit thought about some multicomponent nonlinear integrable model on a lattice suitable for the needs of condensed matter physics has persistently come to mind. We believe a number of physicists have tried to model the electronic (or other intramolecular) subsystem coupled with the subsystem of lattice on-site displacements in terms of some integrable multicomponent model, although undoubtedly a vast majority of other researchers have no illusions on this subject. As an intermediate between these two opposite points of view, we refer to the recent attempt to describe the exciton–vibron system in terms of integrable Ablowitz–Ladik and Toda subsystems coupled, however, with the loss of integrability in the system as a whole [7]. We do not yet know of any reports on a successful coupling of the discrete Schrödinger field and the discrete field of on-site displacements into unified integrable nonlinear system. Moreover, our own attempts to solve this problem have ceased to produce direct results. Nevertheless, the experience of such activity has enabled us to switch our interest exclusively to finding an appropriate coupling between the nonlinear fields of on-site displacements. Even the one-component nonlinear systems of this type now attract considerable attention as the simplest models of granular and structured geophysical media [8–13]. In this respect any new discrete nonlinear system being both integrable and multicomponent deserves to be developed and studied, provided the evolution of each of its components is governed by the second-order derivative with respect to time. We have preferred to realize this intention within the framework of discretized zero-curvature representation [14–19] with the use of a new third-order spectral operator serving to avoid any trivial matrix generalizations of the standard one-component Toda system [20]. Two different ways to increase the order of the spectral operator are described in section 2. One of these, which ideally fitted to both the zero-curvature approach in the generation of nonlinear evolution equations and to the Caudrey treatment of the direct and inverse scattering transforms, is adopted in this paper.

In a series of papers [21–25] Caudrey has developed a fairly constructive version of inverse scattering transform valid in principle for any one-dimensional scattering problem of an arbitrary order. In particular, the method enables us to integrate the nonlinear evolution equations associated with the third-order differential (continuous) and finite difference (discrete) spectral problems in a substantially more simple manner [22, 24] compared with other approaches [26, 27] as well as to treat adequately the so-called loop-like soliton and multisoliton solutions [28, 29].

Bearing in mind the success of the Caudrey approach here we introduce a new third-order discrete spectral problem and show how it generates an integrable dynamical system of three nonlinearly coupled fields on a regular infinite lattice. We give a general sketch of the Caudrey method as applied to the model of interest and test it on the simplest solution.

The main features of the suggested model are as follows. The model describes the three-component nonlinear dynamical system on regular one-dimensional lattice of second order in time with regards to each field variable. Two of its components are mutually equivalent and if required may be combined approximately into the Toda-type field accompanied by some satellite field. The third component of the system is responsible for the angular valued field similar to that in the sine-Gordon model but with another type of nonlinearity and with the standard definitions of spatial and temporal coordinates. The couplings between the components are essentially nonlinear and displayed both in kinetic and potential parts of Lagrangian function.

For the convenience of a formal comparison with our model and its auxiliary spectral problem, we would like to refer to some interesting publications dealing with the discretized integrable nonlinear evolution equations, such as the sine-Gordon lattice model in light-cone coordinates [30–33], the three wave discrete system [34] and a number of other discrete integrable nonlinear systems broadly listed by Tsuchida [35]. This list can be readily extended taking into account appropriate references throughout the main text of our paper.

In section 2 we introduce two auxiliary linear problems in a form suitable to generate the integrable nonlinear model in the framework of the zero-curvature equation with the spectral problem chosen to be of third order. In parallel, we discuss an alternative way of formulating the spectral problem adequate to generate the integrable models within Gel'fand–Dikiy [36], Lax [37, 38] or generalized Wronskian [39] techniques. In appendix A we reformulate our original third-order spectral problem in terms of this alternative approach for the sake of comparison. In appendix B we provide a broad explanation of how the final expressions for the spectral and evolution operators can be obtained.

In section 3 we present the Lagrangian form of the discrete nonlinear dynamical system under study, which has been derived relying upon the zero-curvature approach and we briefly characterize the meaning of its field components.

In sections 4 and 5 we make preparatory steps useful for the Caudrey method to be applied to the integration of our nonlinear model. In particular, we perform the gauge transformation ensuring that the transformed spectral operator possesses the same limits at both spatial infinities (section 4). Then we calculate the eigenvalues of the transformed spectral operator taken at any spatial infinity and find both right and left sets of its eigenfunctions (section 5). These data enable us to divide the plane of the complex spectral parameter into six different regions serving in future as the regions of regularity for the concrete Jost functions.

Section 6 gives the generalized Caudrey definition of (envelope) Jost functions as the solutions of Fredholm (not Volterra) summation equations with meticulously specified kernels. In contrast to the usual definition, the Caudrey definition permits us to avoid a tedious construction procedure of appropriate Jost functions from the solutions of direct and conjugate spectral problems. In a standard definition, such a procedure becomes inevitable beginning already with the third-order spectral problems. Relying upon the formal resolution of Fredholm equations we are able to treat the spectral data in terms of singularities in the plane of the complex spectral parameter emanating in the Jost functions from the singularities of respective resolvents.

In section 7 we observe that the formal mapping of field amplitudes into the Jost functions claimed in section 6 turns out to be incomplete. We correct this mapping via the back gauge transformation in all quantities of interest.

In section 8 we show how to derive the fundamental formulae expressing each residue of one particular Jost function as the superposition of two other Jost functions taken in the point of respective pole location. The set of all superposing coefficients supplemented by the set of all pole locations constitute the discrete part of the scattering data.

In section 9 we formulate the inverse scheme for the reconstruction of (envelope) Jost functions from the discrete part of scattering data. This scheme is valid only for the so-called reflectionless case when the continuous part of scattering data caused by discontinuities of (envelope) Jost functions in the plane of complex spectral parameter is absent. Of course, in general the inverse scattering problem had to be formulated as a Riemann–Hilbert boundary value problem on the lines separating six different regions in the plane of the complex spectral parameter with the presence of poles taken into account. An explicit resolution of such a problem remains beyond the scope of the present investigation in view of expected technical difficulties and due to the fact that it is not necessary for obtaining the purely soliton solutions.

In section 10 the equations governing the temporal evolution of scattering data in reflectionless case are obtained.

In section 11 we establish the one-to-one correspondence between the field amplitudes of our nonlinear model and the temporal derivatives of first expansion coefficients in Laurent-type series of envelope Jost functions.

In section 12, based on the results of previous sections, we obtain the simplest nontrivial solution to the discrete nonlinear dynamical system under study.

In view of their standard role, any descriptions of introductory and concluding sections are omitted.

2. Auxiliary linear problems

In this paper we derive the model of interest starting with the well-known zero-curvature equation [14–19]

$$\frac{\mathrm{d}}{\mathrm{d}\tau}L(n|z) = A(n+1|z)L(n|z) - L(n|z)A(n|z) \tag{1}$$

serving as a compatibility condition for the two auxiliary linear problems

$$|u(n+1|z)\rangle = L(n|z)|u(n|z)\rangle \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}|u(n|z)\rangle = A(n|z)|u(n|z)\rangle \tag{3}$$

with $L(n|z)$, $A(n|z)$ and $|u(n|z)\rangle$ being the spectral matrix operator, the evolution matrix operator and the column matrix vector, respectively. Here n denotes the discrete coordinate variable running through all integers from minus to plus infinity, τ denotes the time variable, while z marks the time-independent spectral parameter. Some authors [14, 16, 17] also call the zero-curvature equation (1) the Lax equation inasmuch as it plays basically a similar role in the procedure of inverse scattering. In any event, the zero-curvature equation (1) permits us both to restore the evolution operator $A(n|z)$ and to generate some nontrivial discrete nonlinear system, provided the spectral operator $L(n|z)$ is appropriately chosen.

The order of the spectral problem (2) is determined by the number of distinct eigenvalues of either the left limiting spectral matrix $L(-\infty|z)$ or the right limiting spectral matrix $L(+\infty|z)$ under the natural premise of both sets of eigenvalues being identical. In this terminology the spectral problems associated with the known multicomponent discrete nonlinear Schrödinger systems [17, 40–43] must be treated as second-order ones despite being rather sophisticated matrix generalizations of the basic Ablowitz–Ladik (i.e. discretized Zakharov–Shabat) spectral problem [4–6]. We also observe similar phenomena in the spectral problems associated with the matrix generalizations of nonlinear Toda systems [44, 45]. In general, however, the higher rank of the spectral matrix $L(n|z)$ could provide a higher order of the respective spectral problem (2).

There is another way to generate new discrete nonlinear integrable systems. It arises from the continuous Gel'fand–Dikiy scheme [36] via formal replacement of differentiation operators in the extended Schrödinger spectral problem into the appropriate powers of the shift operator. We do not discuss here all the features of such an approach and we direct the interested reader to the voluminous literature on this subject; see, for example, [46–48] and references therein. Instead we would like to note that the order of the discretized extended Schrödinger spectral problem with the ordinary potentials is determined by the difference between the highest and the lowest powers of the involved shift operator [49].

Of course, the situation becomes complicated when the matrix-valued discretized extended Schrödinger spectral problem with the matrix-valued potentials is considered [50]. However, as a result we acquire more flexible possibilities in establishing the formal relationship between the spectral problems appearing in different approaches but dealing with the same nonlinear evolution model (see appendix A for a particular example).

Returning to the starting point of our investigation (1)–(3) we adopt $L(n|z)$ and $A(n|z)$ as 3×3 matrices whereas $|u(n|z)\rangle$ is the three-component column vector

$$|u(n|z)\rangle \equiv \begin{pmatrix} \langle 1|u(n|z)\rangle \\ \langle 2|u(n|z)\rangle \\ \langle 3|u(n|z)\rangle \end{pmatrix}. \tag{4}$$

Then taking the spectral operator $L(n|z)$ in the form

$$L(n|z) = \begin{pmatrix} p_{11}(n) + \lambda(z) & F_{12}(n) & p_{13}(n) \\ G_{21}(n) & 0 & G_{23}(n) \\ p_{31}(n) & F_{32}(n) & p_{33}(n) + \lambda(z) \end{pmatrix} \tag{5}$$

and looking for the evolution operator $A(n|z)$ in the form

$$A(n|z) = \begin{pmatrix} 0 & A_{12}(n) & 0 \\ A_{21}(n) & \lambda(z) & A_{23}(n) \\ 0 & A_{32}(n) & 0 \end{pmatrix} \tag{6}$$

we can observe by the direct substitution into the zero-curvature equation (1) that the matrix elements of spectral operator (5) are not independent and should be parametrized in terms of only three independent field variables $q_-(n), \alpha(n), q_+(n)$ and their temporal derivatives (see appendix B for more details). It is worthwhile noting that, in general, such a parametrization may contain two time-independent arbitrary functions of coordinate, which could be responsible for some introduced inhomogeneity of coordinate space [51]. Abstracting from feasible speculations about their role in physical applications we take both above-mentioned functions as sheer constants insofar as even in this simplified case we come to the substantially nontrivial nonlinear dynamical system. Under these conditions and without further loss of generality, the reduction in question can be written as follows

$$F_{12}(n) = \mathrm{i}\exp[+q_-(n)]\cos\alpha(n) \tag{7}$$

$$G_{21}(n) = \mathrm{i}\exp[-q_-(n)]\cos\alpha(n) \tag{8}$$

$$G_{23}(n) = \mathrm{i}\exp[-q_+(n)]\sin\alpha(n) \tag{9}$$

$$F_{32}(n) = \mathrm{i}\exp[+q_+(n)]\sin\alpha(n) \tag{10}$$

$$p_{11}(n) = \dot{q}_-(n)[1 - \sin^4\alpha(n)] - \dot{q}_+(n)\sin^2\alpha(n)\cos^2\alpha(n) \tag{11}$$

$$p_{13}(n) = \exp[+q_-(n) - q_+(n)] \times [\dot{q}_-(n)\sin^3\alpha(n)\cos\alpha(n) + \dot{q}_+(n)\sin\alpha(n)\cos^3\alpha(n) - \dot{\alpha}(n)] \tag{12}$$

$$p_{31}(n) = \exp[-q_-(n) + q_+(n)] \times [\dot{q}_-(n)\sin^3\alpha(n)\cos\alpha(n) + \dot{q}_+(n)\sin\alpha(n)\cos^3\alpha(n) + \dot{\alpha}(n)] \tag{13}$$

$$p_{33}(n) = \dot{q}_+(n)[1 - \cos^4\alpha(n)] - \dot{q}_-(n)\sin^2\alpha(n)\cos^2\alpha(n). \tag{14}$$

Here $q_-(n)$, $q_+(n)$ and $\alpha(n)$ are nothing but the field variables of desired nonlinear evolution model, while the overdot denotes the derivative with respect to time τ. In turn, the matrix elements $A_{jk}(n)$ of the evolution operator $A(n|z)$ are found to be

$$A_{12}(n) = -\mathrm{i}\exp[+q_-(n)]\cos\alpha(n) \tag{15}$$

$$A_{21}(n) = -\mathrm{i}\exp[-q_-(n-1)]\cos\alpha(n-1) \tag{16}$$

$$A_{23}(n) = -\mathrm{i}\exp[-q_+(n-1)]\sin\alpha(n-1) \tag{17}$$

$$A_{32}(n) = -\mathrm{i}\exp[+q_+(n)]\sin\alpha(n). \tag{18}$$

The particular choice of functional dependence $\lambda(z)$ on z in expressions (5) and (6) is dictated by the boundary conditions imposed on the velocities $\dot{q}_+(n), \dot{\alpha}(n), \dot{q}_-(n)$ at both spatial infinities $|n| \to \infty$. Thus, in the case of vanishing limiting velocities $\dot{q}_+(\pm\infty) = 0$, $\dot{\alpha}(\pm\infty) = 0$, $\dot{q}_-(\pm\infty) = 0$ which we use when developing the inverse scattering technique, the choice $\lambda(z) = z + 1/z$ turns out to be the most reasonable.

3. Nonlinear three-component dynamical model

Inserting expressions (7)–(14) and (15)–(18) just obtained into the spectral (5) and evolution (6) matrices respectively, and manipulating with the compatibility (zero-curvature) relation (1), we readily come to the system of three dynamical equations for the field variables $q_+(n)$, $q_-(n)$ and $\alpha(n)$ on an infinite regular one-dimensional chain. For the sake of brevity we prefer to write them in the standard Lagrangian form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}[\partial\mathscr{L}/\partial\dot{q}_+(n)] = \partial\mathscr{L}/\partial q_+(n) \tag{19}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}[\partial\mathscr{L}/\partial\dot{q}_-(n)] = \partial\mathscr{L}/\partial q_-(n) \tag{20}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}[\partial\mathscr{L}/\partial\dot{\alpha}(n)] = \partial\mathscr{L}/\partial \alpha(n) \tag{21}$$

with the Lagrangian function $\mathscr{L}$ given by

$$\begin{aligned}\mathscr{L} = \frac{1}{2}\sum_{m=-\infty}^{\infty}&\left[\dot{q}_+^2(m)\sin^2\alpha(m) + \dot{q}_-^2(m)\cos^2\alpha(m)\right]\\ &- \sum_{m=-\infty}^{\infty}\exp[+q_+(m+1) - q_+(m)]\sin\alpha(m+1)\sin\alpha(m)\\ &- \sum_{m=-\infty}^{\infty}\exp[+q_-(m+1) - q_-(m)]\cos\alpha(m+1)\cos\alpha(m)\\ &+ \frac{1}{2}\sum_{m=-\infty}^{\infty}[\dot{q}_+(m) - \dot{q}_-(m)]^2\sin^2\alpha(m)\cos^2\alpha(m) - \sum_{m=-\infty}^{\infty}\dot{\alpha}^2(m).\end{aligned} \tag{22}$$

The components $q_+(n)$ and $q_-(n)$ are physically equivalent and play a mutually complementary role to each other so that their sum $Q(n) = [q_+(n) + q_-(n)]/2$ can be treated as a Toda-type field while their difference $\rho(n) = [q_+(n) - q_-(n)]/2$ can be thought of as an intrinsic degree of freedom. Apart from its nonlinear wave origin (strictly manifested, for example, in the continuous version of Lagrangian (22)) the field $\alpha(n)$ serves as a coupler of fields $q_+(n)$ and $q_-(n)$ in their kinetic part. This mixture cannot be removed by either diagonalization procedure and hence the very separation of fields $q_+(n)$ and $q_-(n)$ into the Toda $Q(n)$ and intrinsic $\rho(n)$ components appears to be approximate. For this reason we do not abandon the symmetrized dynamical variables $q_+(n)$, $\alpha(n)$, $q_-(n)$ in our subsequent presentation.

According to the general rule, an equivalence between the zero-curvature equation (1) and the nonlinear model of interest (19)–(22) following from the chosen specifications (7)–(14) and (15)–(18) of spectral (5) and evolution (6) matrices opens the door for the model (19)–(22) to be integrated by the method of inverse scattering transform. However, the particular realization of this scheme in our case does not look as simple inasmuch as three (rather than two) distinct eigenvalues constitute the spectrum of the limiting spectral matrix $L(-\infty|z)$ or adequately $L(+\infty|z)$. Therefore, we are bound to rely here upon the Caudrey approach [22, 24] although with some inevitable modifications.

4. Gauge transformed auxiliary linear problems

Inspecting the limits of spectral matrix (5) at both spatial infinities with due regard for adopted reduction (7)–(14) we see that in general they do not coincide

$$\lim_{n\to-\infty} L(n|z) \neq \lim_{n\to+\infty} L(n|z). \tag{23}$$

As a consequence the direct application of Caudrey theory to our problem is formally forbidden.

This type of inconvenience is known also for the Toda system [15] and can in principle be rebuffed by an appropriate gauge transformation

$$|v(n|z)\rangle = S(n)|u(n|z)\rangle \tag{24}$$

$$M(n|z) = S(n+1)L(n|z)S^{-1}(n). \tag{25}$$

The choice of gauge matrix $S(n)$ is not unique and should meet the only basic condition

$$\lim_{n\to-\infty} M(n|z) = M(z) = \lim_{n\to+\infty} M(n|z) \tag{26}$$

where $M(z)$ can be taken as follows

$$M(z) = \begin{pmatrix} z+1/z & F_{12} & 0 \\ G_{21} & 0 & G_{23} \\ 0 & F_{32} & z+1/z \end{pmatrix} \tag{27}$$

$$F_{12} = \mathrm{i}\exp[+q_-]\cos\alpha \tag{28}$$

$$G_{21} = \mathrm{i}\exp[-q_-]\cos\alpha \tag{29}$$

$$G_{23} = \mathrm{i}\exp[-q_+]\sin\alpha \tag{30}$$

$$F_{32} = \mathrm{i}\exp[+q_+]\sin\alpha. \tag{31}$$

For the sake of definiteness we assume the gauge matrix to be

$$S(n) = \begin{pmatrix} S_{11}(n) & 0 & S_{13}(n) \\ 0 & 1 & 0 \\ S_{31}(n) & 0 & S_{33}(n) \end{pmatrix} \tag{32}$$

where

$$S_{11}(n) = \exp[-q_-(n) + q_-]\cos[\alpha(n) - \alpha] \tag{33}$$

$$S_{13}(n) = +\exp[-q_+(n) + q_-]\sin[\alpha(n) - \alpha] \tag{34}$$

$$S_{31}(n) = -\exp[-q_-(n) + q_+]\sin[\alpha(n) - \alpha] \tag{35}$$

$$S_{33}(n) = \exp[-q_+(n) + q_+]\cos[\alpha(n) - \alpha]. \tag{36}$$

The use of the gauge transformed auxiliary linear problems

$$|v(n+1)|z)\rangle = M(n|z)|v(n|z)\rangle \tag{37}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}|v(n|z)\rangle = B(n|z)|v(n|z)\rangle \tag{38}$$

and the gauge transformed zero-curvature relation

$$\dot{M}(n|z) = B(n+1|z)M(n|z) - M(n|z)B(n|z) \tag{39}$$

enables us to remove totally the principal theoretical obstacle described at the beginning of this section. Here $M(n|z)$ and $B(n|z)$ denote the transformed spectral and evolution operators given by equation (25) and

$$B(n|z) = S(n)A(n|z)S^{-1}(n) + \dot{S}(n)S^{-1}(n) \tag{40}$$

respectively.

5. Eigenvalues of $M(z)$ and the domains of their subordination

According to Caudrey [22, 24] the main peculiarities of a particular inverse scattering problem are determined by the spectral properties of limiting spectral matrix $M(z)$. In this context the first step is to resolve the right

$$M(z)|v(z)\rangle = \zeta(z)|v(z)\rangle \tag{41}$$

and the left

$$\langle v^{+}(z)|M(z) = \langle v^{+}(z)|\zeta(z) \tag{42}$$

eigenvalue problems with the limiting gauge transformed spectral operator $M(z)$ given by equations (26)–(31) and to perform the mutual comparison of all eigenvalues

$$\zeta_1(z) = z \tag{43}$$

$$\zeta_2(z) = z + 1/z \tag{44}$$

$$\zeta_3(z) = 1/z \tag{45}$$

with respect to their moduli on the whole plane of complex spectral parameter z.

Thus for the components $\langle k|v_j(z)\rangle$ of right (column) eigenvectors $|v_1(z)\rangle$, $|v_2(z)\rangle$, $|v_3(z)\rangle$ we find

$$\langle 1|v_1(z)\rangle = \mathrm{i}\exp[+q_-]\cos\alpha \tag{46}$$

$$\langle 2|v_1(z)\rangle = -1/z \tag{47}$$

$$\langle 3|v_1(z)\rangle = \mathrm{i}\exp[+q_+]\sin\alpha \tag{48}$$

$$\langle 1|v_2(z)\rangle = +\mathrm{i}\exp[+q_-]\sin\alpha \tag{49}$$

$$\langle 2|v_2(z)\rangle = 0 \tag{50}$$

$$\langle 3|v_2(z)\rangle = -\mathrm{i}\exp[+q_+]\cos\alpha \tag{51}$$

$$\langle 1|v_3(z)\rangle = \mathrm{i}\exp[+q_-]\cos\alpha \tag{52}$$

$$\langle 2|v_3(z)\rangle = -z \tag{53}$$

$$\langle 3|v_3(z)\rangle = \mathrm{i}\exp[+q_+]\sin\alpha. \tag{54}$$

For the components $\langle v_j^{+}(z)|k\rangle$ of left (row) eigenvectors $\langle v_1^{+}(z)|$, $\langle v_2^{+}(z)|$, $\langle v_3^{+}(z)|$ in turn we have

$$\langle v_1^{+}(z)|1\rangle = \mathrm{i}\exp[-q_-]\cos\alpha \tag{55}$$

$$\langle v_1^+(z)|2\rangle = -1/z \tag{56}$$

$$\langle v_1^+(z)|3\rangle = \mathrm{i}\exp[-q_+]\sin\alpha \tag{57}$$

$$\langle v_2^+(z)|1\rangle = -\mathrm{i}\exp[-q_-]\sin\alpha \tag{58}$$

$$\langle v_2^+(z)|2\rangle = 0 \tag{59}$$

$$\langle v_2^+(z)|3\rangle = +\mathrm{i}\exp[-q_+]\cos\alpha \tag{60}$$

$$\langle v_3^+(z)|1\rangle = \mathrm{i}\exp[-q_-]\cos\alpha \tag{61}$$

$$\langle v_3^+(z)|2\rangle = -z \tag{62}$$

$$\langle v_3^+(z)|3\rangle = \mathrm{i}\exp[-q_+]\sin\alpha. \tag{63}$$

The jth eigenvalue $\zeta_j(z)$ pertains equally well both to the jth right $|v_j(z)\rangle$ and jth left $\langle v_j^+(z)|$ eigenvectors. As a result the orthogonality relations

$$\frac{\langle v_j^+(z)|v_k(z)\rangle}{\langle v_j^+(z)|v_j(z)\rangle} = \delta_{jk} \tag{64}$$

are proven to be valid, where

$$\langle v_j^+(z)|v_k(z)\rangle \equiv \sum_{l=1}^{3}\langle v_j^+(z)|l\rangle\langle l|v_k(z)\rangle \tag{65}$$

while j and k run from 1 to 3.

Insofar as all three eigenvalues $\zeta_1(z), \zeta_2(z), \zeta_3(z)$ of $M(z)$ as the functions of z are distinct (see equations (43)–(45)) the complex z plane can inevitably be divided into the six different regions in accordance with the six feasible chains of inequalities between their moduli $|\zeta_1(z)|$, $|\zeta_2(z)|$, $|\zeta_3(z)|$. Speaking formally, the parameter z will be regarded as belonging to the region $D(jkl)$ provided it satisfies the chains of inequalities

$$|\zeta_j(z)| < |\zeta_k(z)| < |\zeta_l(z)| \tag{66}$$

where the sequence $\{jkl\}$ must be detectable among the six possible permutations of sequence $\{123\}$. Due to the evident equalities

$$|\zeta_j(z^*)| = |\zeta_j(z)| = |\zeta_j(-z)| \tag{67}$$

each of regions just defined is subdivided into the two disconnected symmetrical domains either in the top and the bottom quadrants as

$$D(123) = D_t(123) + D_b(123) \tag{68}$$

$$D(213) = D_t(213) + D_b(213) \tag{69}$$

$$D(231) = D_t(231) + D_b(231) \tag{70}$$

$$D(321) = D_t(321) + D_b(321) \tag{71}$$

or predominantly in the left and the right quadrants as

$$D(132) = D_l(132) + D_r(132) \tag{72}$$

$$D(312) = D_l(312) + D_r(312) \tag{73}$$

(see figure 1 for clarity).

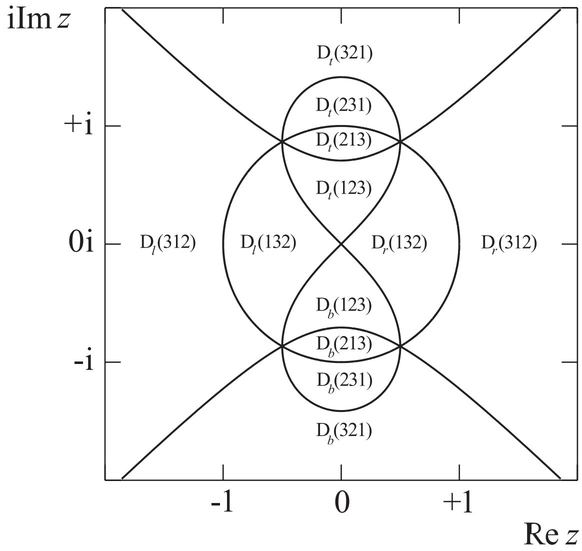

Figure 1. Subdivision into the regions $\{D(jkl)\}$ in the plane of complex spectral parameter z. The parameter z is assumed to belong to $D(jkl)$ if $|\zeta_j(z)| < |\zeta_k(z)| < |\zeta_l(z)|$, where $\zeta_1(z)$, $\zeta_2(z)$, $\zeta_3(z)$ are the eigenvalues of $M(z)$. The pairs of subindices l, r and t, b are written to distinguish two disconnected parts (domains) of the same region, e.g. $D(132) = D_l(132) + D_r(132)$.

6. Direct scattering problem: the advanced Caudrey approach

The advanced version of Caudrey approach to the direct and inverse scattering problems is based upon his generalized definition of Jost functions [22, 24] which, in contrast to the standard ones [21, 23, 26, 27], succeeds in avoiding any address to the conjugate spectral problem even in the highly complicated cases of spectral plane subdivision. As a result, both the direct and the inverse scattering theories acquire the forms of essentially formalized procedure weakly sensitive to the particular order of the spectral operator.

In this section, we will substantially use the definitions and results of the previous two sections. Moreover, we will operate with the nonstandard term 'jth envelope Jost function' serving for the jth Jost function multiplied by the factor $[\zeta_j(z)]^{-n}$ and hence demonstrating the more smooth dependence on spatial coordinate n.

Following Caudrey [22, 24] we adopt the jth envelope Jost function

$$|\Phi_j(n|z)\rangle = [\zeta_j(z)]^{-n}|\varphi_j(n|z)\rangle \qquad (j = 1, 2, 3) \tag{74}$$

associated with the gauge transformed auxiliary spectral problem (37) as a solution to the jth Fredholm summation equation

$$|\Phi_j(n|z)\rangle = |v_j(z)\rangle + \sum_{m=-\infty}^{\infty} K_j(n|z|m)|\Phi_j(m|z)\rangle \qquad (j = 1, 2, 3) \tag{75}$$

with the jth kernel matrix $K_j(n|z|m)$ specified by the expression

$$K_j(n|z|m) = [\zeta_j(z)]^{m-n}[M(z)]^{n-m-1}\left[\theta(n-m)I - \sum_{k=1}^{3}\theta(|\zeta_k(z)| - |\zeta_j(z)|)P_k(z)\right] \times [M(m|z) - M(z)] \qquad (j = 1, 2, 3) \tag{76}$$

where $P_k(z)$ denotes the kth projection operator

$$P_k(z) \equiv \frac{|v_k(z)\rangle\langle v_k^+(z)|}{\langle v_k^+(z)|v_k(z)\rangle} \qquad (k = 1, 2, 3) \tag{77}$$

I denotes the unity 3×3 matrix and

$$\theta(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leqslant 0. \end{cases} \tag{78}$$

We call $|\varphi_j(n|z)\rangle$ the jth Jost function inasmuch as, according to the relation (74) and the jth Fredholm equation (75), it meets all the necessary demands of the usual definition, i.e. it satisfies both the gauge transformed spectral equation (37) and the standard boundary condition

$$\lim_{n\to-\infty}[\zeta_j(z)]^{-n}|\varphi_j(n|z)\rangle = |v_j(z)\rangle \qquad (j = 1, 2, 3). \tag{79}$$

An extra property affirming the boundedness of each envelope Jost function $|\Phi_j(n|z)\rangle$ $(j = 1, 2, 3)$ at $n \to +\infty$ follows from the respective Fredholm equation (75) after the use of identity

$$\begin{aligned} \theta(n-m)\cdot I - \sum_{k=1}^{3}\theta\left(|\zeta_k(z)| - |\zeta_j(z)|\right) P_k(z) \\ \equiv \sum_{k=1}^{3}\theta(|\zeta_j(z)| - |\zeta_k(z)|)P_k(z) - \theta(m+1-n)\cdot I + P_j(z) \\ + \sum_{k=1}^{3}(1-\delta_{jk})[1 - \theta(|\zeta_k(z)| - |\zeta_j(z)|) - \theta(|\zeta_j(z)| - |\zeta_k(z)|)]P_k(z) \end{aligned} \tag{80}$$

in the respective kernel matrix (76).

When proving the boundary properties of Jost functions at $n \to -\infty$ and $n \to +\infty$, an assumption about the sufficiently rapid decrease of $M(n|z) - M(z)$ to the zero 3×3 matrix at respective infinity of coordinate n turns out to be rather convenient.

Actually the same assumption about the rapid vanishing of $M(n|z) - M(z)$ at $|n| \to \infty$ supports a sufficient condition for the Fredholm determinant $f_j(z)$ and the 3×3 matrix analogue of the first Fredholm minor $F_j(n|z|m)$ $(j = 1, 2, 3)$ associated with the respective Fredholm equation (75) to exist in every spectral region defined in section 5. Here it should be particularly emphasized that our spectral parameter z has nothing to do with the auxiliary parameter of standard Fredholm theory [52]. Moreover, each kernel matrix $K_j(n|z|m)$ $(j = 1, 2, 3)$ is seen (equation (76)) to be a piecewise function of spectral parameter, i.e. to exhibit a jump discontinuity once the parameter z crosses the boundary between the spectral domains. In principle, the similar discontinuities may be displayed also in $f_j(z)$ and $F_j(n|z|m)$ $(j = 1, 2, 3)$ inasmuch as these quantities are linked to the respective $K_j(n|z|m)$ through the basic Fredholm relations

$$F_j(n|z|m) - f_j(z)K_j(n|z|m) = \sum_{l=-\infty}^{\infty} F_j(n|z|l)K_j(l|z|m) \qquad (j = 1, 2, 3) \tag{81}$$

$$F_j(n|z|m) - f_j(z)K_j(n|z|m) = \sum_{l=-\infty}^{\infty} K_j(n|z|l)F_j(l|z|m) \qquad (j = 1, 2, 3). \tag{82}$$

However, within the interior of each individual spectral domain the quantities $f_j(z)$ and $F_j(n|z|m)$ $(j = 1, 2, 3)$ have to be regular functions of spectral parameter z [53] insofar as the same is certainly true for the respective kernel $K_j(n|z|m)$ (equation (76)). The limits of these functions as z approaches any interdomain boundary exist and are finite.

Thus, at all z providing $f_j(z) \neq 0$ the jth 3×3 matrix Fredholm resolvent

$$R_j(n|z|m) = \frac{F_j(n|z|m)}{f_j(z)} \qquad (j = 1, 2, 3) \tag{83}$$

is defined and the formal solution of jth Fredholm equation (75) is given by

$$|\Phi_j(n|z)\rangle = |v_j(z)\rangle + \sum_{m=-\infty}^{\infty} R_j(n|z|m)|v_j(z)\rangle \qquad (j = 1, 2, 3). \tag{84}$$

We can reckon that this result (84) for the set of envelope Jost functions $\{|\Phi_j(n|z)\rangle\}$ is the main step in mapping the set of dynamical variables $\{q_+(n), \alpha(n), q_-(n)\}$ into the set of scattering data. The latter in Caudrey terminology [21–24] is nothing but the information about the generic poles (i.e. the poles unremovable by any renormalization of the set $\{|v_j(z)\rangle\}$) of Jost functions and their residues as well as about possible discontinuities of Jost functions in the complex z plane. Evidently these poles are determined by the zeros of Fredholm determinants while the jump singularities may occur only on the boundaries between the domains in the plane of complex spectral parameter (i.e. spectral domains).

7. Direct scattering transform: how to achieve an explicit mapping

Strictly speaking, the mapping from the field amplitudes into the Jost functions given in its present form (84) is fairly implicit insofar as the field amplitudes $q_-(n)$, $\alpha(n)$, $q_+(n)$ remain explicitly traceable in all Jost functions under consideration. Unfortunately this fact becomes evident only in the final stage of inversion from the scattering data to the Jost functions and can be plainly elucidated, for example, in an extra property

$$\lim_{|\zeta_j(z)|\to\infty} [|\Phi_j(n|z)\rangle - S(n)S^{-1}(-\infty)|v_j(z)\rangle] = 0 \cdot I \qquad (j = 1, 2, 3) \tag{85}$$

necessary for the inversion to be fixed uniquely.

Looking at the fixing condition (85) we find that the situation with an implicit mapping can be readily handled via the simple back gauge transformation in all quantities of interest. Thus, instead of the jth Jost function $|\varphi_j(n|z)\rangle$ and jth envelope Jost function $|\Phi_j(n|z)\rangle$ $(j = 1, 2, 3)$ we have to use their back gauge transforms

$$|\xi_j(n|z)\rangle = S^{-1}(n)|\varphi_j(n|z)\rangle \qquad (j = 1, 2, 3) \tag{86}$$

and

$$|\Xi_j(n|z)\rangle = S^{-1}(n)|\Phi_j(n|z)\rangle \qquad (j = 1, 2, 3). \tag{87}$$

accordingly. The jth resolvent solution (84) in turn is apparently converted to yield

$$|\Xi_j(n|z)\rangle = |u_j(z)\rangle + \sum_{m=-\infty}^{\infty} D_j(n|z|m)|u_j(z)\rangle \qquad (j = 1, 2, 3) \tag{88}$$

where

$$|u_j(z)\rangle = S^{-1}(-\infty)|v_j(z)\rangle \qquad (j = 1, 2, 3) \tag{89}$$

denotes the jth right eigenvector of operator

$$L(z) = S^{-1}(-\infty)M(z)S(-\infty) = L(-\infty|z) \tag{90}$$

and

$$D_j(n|z|m) = S^{-1}(n)R_j(n|z|m)S(-\infty) + [S^{-1}(n)S(-\infty) - I]\delta_{nm} \qquad (j = 1, 2, 3) \tag{91}$$

denotes the jth back transformed resolvent. Finally, the jth fixing condition (85) is rewritten as

$$\lim_{|\zeta_j(z)|\to\infty}[|\Xi_j(n|z)\rangle - |u_j(z)\rangle] = 0\cdot I \qquad (j = 1, 2, 3). \tag{92}$$

It is worth noticing that, while making the mapping explicit, the back gauge transform saves all fundamental properties of Jost functions and envelope Jost functions to be carried over into their back gauge transformed counterparts. Therefore, it looks reasonable to treat $|\xi_j(n|z)\rangle$ and $|\Xi_j(n|z)\rangle$ as the jth Jost function and the jth envelope Jost function ($j = 1, 2, 3$) associated directly with the original auxiliary spectral problem (2).

8. Scattering data: reflectionless case

According to the general rule [22, 24], the scattering data can be identified with information about the singularities of resolvent matrices. In this paper, we adopt such a definition and consider the so-called reflectionless case when the jump singularities of Jost functions on boundaries between the domains in the plane of the complex spectral parameter are absent and the only informative singularities of Jost functions are the poles emanated from zeros of respective Fredholm determinants.

In order to examine these poles we have to know the Wronskian

$$\mathop{\mathrm{W}}_{k=1}^{3}\{|\xi_k(n|z)\rangle\} \equiv \det[\langle j|\xi_k(n|z)\rangle] \tag{93}$$

calculated on the Jost solutions of original spectral problem (2). Applying the Wronskian operation (93) to the set of equalities

$$|\xi_k(n+1|z)\rangle = L(n|z)|\xi_k(n|z)\rangle \qquad (k = 1, 2, 3) \tag{94}$$

with the use of asymptotic properties

$$\lim_{n\to-\infty}[\zeta_k(z)]^{-n}|\xi_k(n|z)\rangle = |u_k(z)\rangle \qquad (k = 1, 2, 3) \tag{95}$$

and equalities

$$L(z)|u_k(z)\rangle = \zeta_k(z)|u_k(z)\rangle \qquad (k = 1, 2, 3) \tag{96}$$

we obtain

$$\mathop{\mathrm{W}}_{k=1}^{3}\{|\xi_k(n|z)\rangle\} = \mathop{\mathrm{W}}_{k=1}^{3}\{|u_k(z)\rangle\}[\det L(z)]^n \prod_{m=-\infty}^{n-1}\left\{\frac{\det L(m|z)}{\det L(z)}\right\}. \tag{97}$$

Now we can readily conclude that the right-hand side of expression (97) does not contain any specific information about the singularities of resolvent matrix (91) because the same is true separately for its cofactors

$$\mathop{\mathrm{W}}_{k=1}^{3}\{|u_k(z)\rangle\} = (z - 1/z)\exp[+q_+(-\infty) + q_-(-\infty)] \tag{98}$$

and

$$\det L(z) = \det L(m|z) = z + 1/z. \tag{99}$$

Thus we have

$$\mathop{\mathrm{W}}_{k=1}^{3}\{\xi_k(n|z)\rangle\} = [z + 1/z]^n[z - 1/z]\exp[+q_+(-\infty) + q_-(-\infty)]. \tag{100}$$

To proceed further, we denote $z_j(r)$ to be the rth zero of the jth Fredholm determinant $f_j(z)$ and we assume that each zero is simple, i.e.

$$f_j(z_j(r)) = 0 \qquad \lim_{z\to z_j(r)} \mathrm{d}f_j(z)/\mathrm{d}z \neq 0 \qquad (r = 1, 2, 3, \ldots, N_j;\ j = 1, 2, 3) \tag{101}$$

and does not lie on any boundary between the spectral domains. Then the rth residue of the jth Jost function $|\xi_j(n|z)\rangle$ can be defined in a manner

$$|\mathrm{Res}[\xi_j(n|z_j(r))]\rangle \equiv \lim_{z\to z_j(r)} \{|\xi_j(n|z)\rangle[z - z_j(r)]\} \qquad (r = 1, 2, 3, \ldots, N_j;\ j = 1, 2, 3) \tag{102}$$

usual for the simple poles. Also, to ensure the finiteness of residues, we have to adopt zeros belonging to different Fredholm determinants to be distinct.

This latter demand becomes natural when inspecting the set of relationships

$$|\mathrm{Res}[\xi_j(n|z_j(r))]\rangle = \lim_{z\to z_j(r)} \sum_{k=1}^{3} |\xi_k(n|z)\rangle\gamma_{kj}(r)[1 - \delta_{jk}] \qquad (r = 1, 2, 3, \ldots, N_j;\ j = 1, 2, 3) \tag{103}$$

where the coefficients $\gamma_{kj}(r)$ $(r = 1, 2, 3, \ldots, N_j;\ j = 1, 2, 3;\ k \neq j)$ and the locations of poles $z_j(r)$ $(r = 1, 2, 3, \ldots, N_j;\ j = 1, 2, 3)$ are referred to as a discrete part of scattering data [22, 24]. Each of the $N_1 + N_2 + N_3$ relationships (103) represents the condition of linear dependence between the columns of a certain 3×3 matrix possessing the zero-valued determinant, while the very set of zero-valued determinants

$$\mathop{\mathrm{W}}_{k=1}^{3}\{|\mathrm{Res}[\xi_j(n|z_j(r))]\rangle\delta_{jk} + \lim_{z\to z_j(r)} |\xi_k(n|z)\rangle[1 - \delta_{jk}]\} = 0 \qquad (r = 1, 2, 3, \ldots, N_j;\ j = 1, 2, 3) \tag{104}$$

arises from the basic Wronskian (100) as a consequence of the simple limiting operation $\lim_{z\to z_j(r)}\{\ldots[z - z_j(r)]\}$ taken separately for each of the $N_1 + N_2 + N_3$ poles. The weak point in the suggested arguments lies in the tacitly supposed coordinate independence of the superposing coefficients $\gamma_{kj}(r)$. However, it is precisely this hypothesis which proves to be the only plausible assumption giving rise to the self-consistent time evolution of scattering data, thus ensuring the noncontradictive character of the whole theory.

Another concretization of superposing coefficients

$$\gamma_{kj}(r) = \theta(|\zeta_k(z_j(r))| - |\zeta_j(z_j(r))|)\Gamma_{kj}(r) \qquad (r = 1, 2, 3, \ldots, N_j;\ j = 1, 2, 3;\ k \neq j) \tag{105}$$

is a direct consequence of asymptotic conditions

$$\lim_{n\to-\infty} |\Xi_j(n|z)\rangle = |u_j(z)\rangle \qquad (j = 1, 2, 3) \tag{106}$$

which can be easily observed during the reconstruction of the envelope Jost functions.

Additional information about the scattering data can be obtained by the symmetry analysis of envelope Jost functions and will be taken into account in the final formulae of their reconstruction. The symmetries of interest may be formally divided into two groups

$$|\Xi_1(n|1/z)\rangle = |\Xi_3(n|z)\rangle \tag{107}$$

$$|\Xi_2(n|1/z)\rangle = |\Xi_2(n|z)\rangle \tag{108}$$

$$|\Xi_3(n|1/z)\rangle = |\Xi_1(n|z)\rangle \tag{109}$$

and

$$\langle k|\Xi_j(n|z^*)\rangle^* = (-1)^k \langle k|\Xi_j(n|z)\rangle \qquad (j = 1, 2, 3;\ k = 1, 2, 3). \tag{110}$$

While the first group (107)–(109) becomes evident from the very forms of the original spectral operator (5) and the limiting eigenfunctions (89) written explicitly, the second (110) should invoke an extra assumption about the reality of dynamical variables $q_-(n)$, $\alpha(n)$, $q_+(n)$ for its proper justification.

9. Reconstruction of envelope Jost functions in terms of their residues

Relying upon the evident correspondence

$$|\xi_j(n|z)\rangle = [\zeta_j(z)]^n |\Xi_j(n|z)\rangle \qquad (j = 1, 2, 3) \tag{111}$$

between the Jost functions $|\xi_j(n|z)\rangle$ and the envelope Jost functions $|\Xi_j(n|z)\rangle$ and using the fundamental formulae (103) for the residues $|\mathrm{Res}[\xi_j(n|z_j(r))]\rangle$ of Jost functions concretized by the expressions (105) we readily come to the similar formulae

$$\begin{aligned}|\mathrm{Res}[\Xi_j(n|z_j(r))]\rangle = {}& \lim_{z\to z_j(r)} \sum_{k=1}^{3} |\Xi_k(n|z)\rangle \\ & \times \theta(|\zeta_k(z_j(r))| - |\zeta_j(z_j(r))|)\Gamma_{kj}(r)[1-\delta_{jk}]\left[\frac{\zeta_k(z_j(r))}{\zeta_j(z_j(r))}\right]^n \\ & (r = 1, 2, 3, \ldots, N_j;\ j = 1, 2, 3)\end{aligned} \tag{112}$$

for the residues

$$\begin{aligned}&|\mathrm{Res}[\Xi_j(n|z_j(r))]\rangle \equiv \lim_{z\to z_j(r)} \{|\Xi_j(n|z)\rangle[z - z_j(r)]\} \\ &(r = 1, 2, 3, \ldots, N_j;\ j = 1, 2, 3)\end{aligned} \tag{113}$$

of envelope Jost functions.

Now we have collected all necessary information (92), (106), (107)–(109), (110), (112) sufficient to reconstruct the general features of envelope Jost functions in the reflectionless case. The methods of complex variable theory yield

$$\begin{aligned}|\Xi_1(n|z)\rangle = {}& |u_1(z)\rangle + (1-\delta_{0N})\sum_{r=1}^{N} |\Xi_2(n|z_1(r))\rangle C_{21}(n|r)\frac{\zeta_1(z_1(r))}{\zeta_1(z) - \zeta_1(z_1(r))} \\ & + (1-\delta_{0N})\sum_{r=1}^{N} |\Xi_3(n|z_1(r))\rangle C_{31}(n|r)\frac{\zeta_1(z_1(r))}{\zeta_1(z) - \zeta_1(z_1(r))} \\ & + (1-\delta_{0N})\sum_{r=1}^{N} |\Xi_2(n|z_1^*(r))\rangle C_{21}^*(n|r)\frac{\zeta_1(z_1^*(r))}{\zeta_1(z) - \zeta_1(z_1^*(r))} \\ & + (1-\delta_{0N})\sum_{r=1}^{N} |\Xi_3(n|z_1^*(r))\rangle C_{31}^*(n|r)\frac{\zeta_1(z_1^*(r))}{\zeta_1(z) - \zeta_1(z_1^*(r))}\end{aligned} \tag{114}$$

$$|\Xi_2(n|z)\rangle = |u_2(z)\rangle + (1-\delta_{0M})\sum_{r=1}^{M}|\Xi_1(n|\bar{z}_2(r))\rangle\,\bar{C}_{12}(n|r)\frac{\zeta_2(\bar{z}_2(r))}{\zeta_2(z)-\zeta_2(\bar{z}_2(r))}$$
$$+(1-\delta_{0M})\sum_{r=1}^{M}|\Xi_3(n|\bar{z}_2(r))\rangle\,\bar{C}_{32}(n|r)\frac{\zeta_2(\bar{z}_2(r))}{\zeta_2(z)-\zeta_2(\bar{z}_2(r))}$$
$$+(1-\delta_{0M})\sum_{r=1}^{M}|\Xi_1(n|\bar{z}_2^*(r))\rangle\,\bar{C}_{12}^*(n|r)\frac{\zeta_2(\bar{z}_2^*(r))}{\zeta_2(z)-\zeta_2(\bar{z}_2^*(r))}$$
$$+(1-\delta_{0M})\sum_{r=1}^{M}|\Xi_3(n|\bar{z}_2^*(r))\rangle\,\bar{C}_{32}^*(n|r)\frac{\zeta_2(\bar{z}_2^*(r))}{\zeta_2(z)-\zeta_2(\bar{z}_2^*(r))} \qquad (115)$$

$$|\Xi_2(n|z)\rangle = |u_2(z)\rangle + (1-\delta_{0M})\sum_{r=1}^{M}|\Xi_3(n|\overset{+}{z}_2(r))\rangle\,\overset{+}{C}_{32}(n|r)\frac{\zeta_2(\overset{+}{z}_2(r))}{\zeta_2(z)-\zeta_2(\overset{+}{z}_2(r))}$$
$$+(1-\delta_{0M})\sum_{r=1}^{M}|\Xi_1(n|\overset{+}{z}_2(r))\rangle\,\overset{+}{C}_{12}(n|r)\frac{\zeta_2(\overset{+}{z}_2(r))}{\zeta_2(z)-\zeta_2(\overset{+}{z}_2(r))}$$
$$+(1-\delta_{0M})\sum_{r=1}^{M}|\Xi_3(n|\overset{+}{z}{}_2^*(r))\rangle\,\overset{+}{C}{}_{32}^*(n|r)\frac{\zeta_2(\overset{+}{z}{}_2^*(r))}{\zeta_2(z)-\zeta_2(\overset{+}{z}{}_2^*(r))}$$
$$+(1-\delta_{0M})\sum_{r=1}^{M}|\Xi_1(n|\overset{+}{z}{}_2^*(r))\rangle\,\overset{+}{C}{}_{12}^*(n|r)\frac{\zeta_2(\overset{+}{z}{}_2^*(r))}{\zeta_2(z)-\zeta_2(\overset{+}{z}{}_2^*(r))} \qquad (116)$$

$$|\Xi_3(n|z)\rangle = |u_3(z)\rangle + (1-\delta_{0N})\sum_{r=1}^{N}|\Xi_1(n|z_3(r))\rangle C_{13}(n|r)\frac{\zeta_3(z_3(r))}{\zeta_3(z)-\zeta_3(z_3(r))}$$
$$+(1-\delta_{0N})\sum_{r=1}^{N}|\Xi_2(n|z_3(r))\rangle C_{23}(n|r)\frac{\zeta_3(z_3(r))}{\zeta_3(z)-\zeta_3(z_3(r))}$$
$$+(1-\delta_{0N})\sum_{r=1}^{N}|\Xi_1(n|z_3^*(r))\rangle C_{13}^*(n|r)\frac{\zeta_3(z_3^*(r))}{\zeta_3(z)-\zeta_3(z_3^*(r))}$$
$$+(1-\delta_{0N})\sum_{r=1}^{N}|\Xi_2(n|z_3^*(r))\rangle C_{23}^*(n|r)\frac{\zeta_3(z_3^*(r))}{\zeta_3(z)-\zeta_3(z_3^*(r))} \qquad (117)$$

where

$$C_{k1}(n|r) = \theta(|\zeta_k(z_1(r))| - |\zeta_1(z_1(r))|)[\zeta_k(z_1(r))/\zeta_1(z_1(r))]^n C_{k1}(r)$$
$$(r = 1, 2, 3, \ldots, N;\ k = 2, 3;\ z_1(r)z_3(r) = 1) \qquad (118)$$

$$\bar{C}_{k2}(n|r) = \theta(|\zeta_k(\bar{z}_2(r))| - |\zeta_2(\bar{z}_2(r))|)[\zeta_k(\bar{z}_2(r))/\zeta_2(\bar{z}_2(r))]^n\,\bar{C}_{k2}(r)$$
$$(r = 1, 2, 3, \ldots, M;\ k = 1, 3;\ \bar{z}_2(r)\,\overset{+}{z}_2(r) = 1) \qquad (119)$$

$$\overset{+}{C}_{k2}(n|r) = \theta(|\zeta_k(\overset{+}{z}_2(r))| - |\zeta_2(\overset{+}{z}_2(r))|)[\zeta_k(\overset{+}{z}_2(r))/\zeta_2(\overset{+}{z}_2(r))]^n\,\overset{+}{C}_{k2}(r)$$
$$(r = 1, 2, 3, \ldots, M;\ k = 3, 1;\ \overset{+}{z}_2(r)\,\bar{z}_2(r) = 1) \qquad (120)$$

$$C_{k3}(n|r) = \theta(|\zeta_k(z_3(r))| - |\zeta_3(z_3(r))|)[\zeta_k(z_3(r))/\zeta_3(z_3(r))]^n C_{k3}(r)$$
$$(r = 1, 2, 3, \ldots, N;\ k = 1, 2;\ z_3(r)z_1(r) = 1). \qquad (121)$$

Here we have introduced the new numeration of poles originated from the manifested symmetries (107)–(110) of envelope Jost functions. In so doing we had to assume that $N_1 = 2N = N_3$, $N_2 = 4M$ and

$$C_{21}(r) = C_{23}(r) \qquad (r = 1, 2, 3, \ldots, N) \tag{122}$$

$$C_{31}(r) = C_{13}(r) \qquad (r = 1, 2, 3, \ldots, N) \tag{123}$$

$$\overset{-}{C}_{12}(r) = \overset{+}{C}_{32}(r) \qquad (r = 1, 2, 3, \ldots, M) \tag{124}$$

$$\overset{-}{C}_{32}(r) = \overset{+}{C}_{12}(r) \qquad (r = 1, 2, 3, \ldots, M). \tag{125}$$

It is convenient to treat the quantities $C_{21}(r), C_{31}(r), z_1(r), C_{23}(r), C_{13}(r), z_3(r)$ (where $r = 1, 2, 3, \ldots, N$) and $\overset{-}{C}_{12}(r), \overset{-}{C}_{32}(r), \overset{-}{z}_2(r), \overset{+}{C}_{32}(r), \overset{+}{C}_{12}(r), \overset{+}{z}_2(r)$ (where $r = 1, 2, 3, \ldots, M$) together with their complex conjugate counterparts as the new more adequate although overfilled set of scattering data rather than to decipher them in terms of the original set.

The general scheme of complete reconstruction of envelope Jost functions from the scattering data in the reflectionless case becomes clear, since it is reduced actually to resolving the set of linear algebraic equations with respect to quantities $|\Xi_j(n|z_1(r))\rangle, |\Xi_j(n|z_1^*(r))\rangle$ (where $j = 2, 3; r = 1, 2, 3, \ldots, N$) and $|\Xi_j(n|\overset{-}{z}_2(r))\rangle, |\Xi_j(n|\overset{-}{z}{}_2^*(r))\rangle$ (where $j = 1, 3; r = 1, 2, 3, \ldots, M$) or alternatively with respect to quantities $|\Xi_j(n|\overset{+}{z}_2(r))\rangle, |\Xi_j(\overset{+}{z}{}_2^*(r))\rangle$ (where $j = 3, 1; r = 1, 2, 3, \ldots, M$) and $|\Xi_j(n|z_3(r))\rangle, |\Xi_j(n|z_3^*(r))\rangle$ (where $j = 1, 2; r = 1, 2, 3, \ldots, N$). However its practical implementation is determined by the additional condition (173) supporting the self-consistency of field variables through the interdependency of scattering data (see section 11 for more details).

In formulae (114)–(117) we have introduced the factors $(1 - \delta_{0N})$ and $(1 - \delta_{0M})$ in order to exclude any confusion, provided N or M or both N and M are equal to zero. In the latter case, no poles are present and the reconstruction of envelope Jost functions is trivialized giving rise to the respective trivialization in the solution of the basic nonlinear model. This conclusion is valid only on the class of reflectionless potentials, i.e. potentials $q_+(n), \alpha(n), q_-(n), \dot{q}_+(n), \dot{\alpha}(n), \dot{q}_-(n)$ usually assumed to be reflectionless at the initial moment $\tau = 0$ and persisting in being reflectionless at any arbitrary time τ. However, the reflectionless potentials are precisely the potentials realizing the soliton solutions of original nonlinear model (19)–(22) which usually are the most interesting solutions for physical applications.

In the case of reflecting potentials, the equations for the reconstruction of envelope Jost functions must include not only the terms responsible for the discrete spectrum (as in equations (114)–(117)) but the additional terms responsible for the continuous spectrum and arising formally from the jump singularities of envelope Jost functions on the cuts along the lines dividing the plane of the complex spectral parameter z into the domains (see figure 1). These additional terms contain the Cauchy-type integrals from the envelope Jost functions taken along the boundaries between domains (see, for example, [22, 24] for the general theory) and make the problem of complete reconstruction of envelope Jost functions highly complicated insofar as it becomes equivalent to the combined set of Fredholm integral equations and linear algebraic equations. In terms of the Marchenko approach [54] such a problem corresponds to the Marchenko-type summation equation with the nondegenerated kernel and there is no simple algorithm for its *explicit* resolution. This is the main reason why the case of reflecting potentials remains as a rule less studied, although for the sake of truth the attempt of finding explicit solutions of a second-order discrete inverse scattering problem that are not restricted to the pure soliton case has recently been made [55] (see also [33]).

10. Time dependences of scattering data

Below, we derive the evolution equations for the scattering data as they were defined in the previous section.

As a first step, it is reasonable to suppose that $\dot{M}(z) = 0 \cdot I$. Then at both spatial infinities $|n| \to \infty$ the gauge transformed zero-curvature relation (39) yields

$$B(z)M(z) = M(z)B(z) \tag{126}$$

where the limiting value

$$B(z) = \lim_{|n|\to\infty} B(n|z) \tag{127}$$

of gauge transformed evolution operator (40) is given by

$$B(z) = (z + 1/z) \cdot I - M(z). \tag{128}$$

Thus the operators $B(z)$ and $M(z)$ are commutative and hence possess the same two sets of eigenvectors, namely the right (46)–(54) and the left (55)–(63) ones. However, the eigenvalues

$$\eta_1(z) = 1/z \tag{129}$$

$$\eta_2(z) = 0 \tag{130}$$

$$\eta_3(z) = z \tag{131}$$

of $B(z)$ and eigenvalues (43)–(45) of $M(z)$ are seen to be distinct. Of course, the jth eigenvalue $\eta_j(z)$ pertains equally well both to the jth right $|v_j(z)\rangle$ and jth left $\langle v_j^+(z)|$ eigenvectors, i.e.

$$B(z)|v_j(z)\rangle = \eta_j(z)|v_j(z)\rangle \qquad (j = 1, 2, 3) \tag{132}$$

$$\langle v_j^+(z)|B(z) = \langle v_j^+(z)|\eta_j(z) \qquad (j = 1, 2, 3). \tag{133}$$

Furthermore, in parallel with the right Jost functions $|\varphi_j(n|z)\rangle$ satisfying

$$M(n|z)|\varphi_j(n|z)\rangle = |\varphi_j(n+1|z)\rangle \qquad (j = 1, 2, 3) \tag{134}$$

$$\lim_{n\to-\infty} [\zeta_j(z)]^{-n}|\varphi_j(n|z)\rangle = |v_j(z)\rangle \qquad (j = 1, 2, 3) \tag{135}$$

we will take advantage of the left Jost functions $\langle\varphi_j^+(n|z)|$ satisfying

$$\langle\varphi_j^+(n+1|z)|M(n|z) = \langle\varphi_j^+(n|z)| \qquad (j = 1, 2, 3) \tag{136}$$

$$\lim_{n\to-\infty} \langle\varphi_j^+(n|z)|[\zeta_j(z)]^n = \langle v_j^+(z)| \qquad (j = 1, 2, 3). \tag{137}$$

The property of orthogonality

$$\langle\varphi_j^+(n|z)|\varphi_k(n|z)\rangle = \langle v_j^+(z)|v_k(z)\rangle = \langle v_j^+(z)|v_j(z)\rangle\delta_{jk} \qquad (j = 1, 2, 3;\ k = 1, 2, 3) \tag{138}$$

will also be necessary.

The results of the previous two paragraphs enable us to prove that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}|\varphi_j(n|z)\rangle = [B(n|z) - \eta_j(z) \cdot I]|\varphi_j(n|z)\rangle \qquad (j = 1, 2, 3). \tag{139}$$

Indeed, differentiating equation (134) with respect to time τ with the subsequent substitute of $\dot{M}(n|z)$ from the gauge transformed zero-curvature relation (39) we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}|\varphi_j(n|z)\rangle - B(n|z)|\varphi_j(n|z)\rangle = \sum_{k=1}^{3} |\varphi_k(n|z)\rangle d_{kj}(z) \qquad (j = 1, 2, 3) \tag{140}$$

insofar as the left-hand side of equation (140) was revealed to satisfy to the gauge transformed spectral equation (37). Then operating on to equation (140) with $\langle\varphi_i^+(n|z)|$ from left to right and taking the limit at $n \to -\infty$ we obtain

$$d_{ij}(z) = -\eta_i(z)\delta_{ij} \qquad (i = 1, 2, 3;\ j = 1, 2, 3) \tag{141}$$

which in combination with equation (140) yields equation (139).

In terms of envelope Jost functions we evidently have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}|\Phi_j(n|z)\rangle = [B(n|z) - \eta_j(z)\cdot I]|\Phi_j(n|z)\rangle \qquad (j = 1, 2, 3). \tag{142}$$

The simple manipulation with equations (127), (133) and (142) gives rise to

$$\lim_{n\to+\infty} \frac{\mathrm{d}}{\mathrm{d}\tau}\langle v_j^+(z)|\Phi_j(n|z)\rangle = 0 \qquad (j = 1, 2, 3). \tag{143}$$

This property (143) together with those written down earlier, equations (127), (133) and (142), comprise the main tool in establishing the evolution equations for the scattering data. In so doing we have to prepare the reconstructions for $\langle v_j^+(z)|\Phi_j(n|z)\rangle (j = 1, 2, 3)$ from the reconstructions (114)–(117) for $|\Xi_j(n|z)\rangle$ $(j = 1, 2, 3)$ by means of gauge transformation

$$|\Phi_j(n|z)\rangle = S(n)|\Xi_j(n|z)\rangle \qquad (j = 1, 2, 3) \tag{144}$$

and then to make two tricks assuming that $\mathrm{d}[S(\infty)S^{-1}(-\infty)]/\mathrm{d}\tau = 0$.

The first consists of calculating

$$\lim_{n\to+\infty} \lim_{\zeta_j(z)\to\zeta_j(z_j(s))} \left\{[\zeta_j(z) - \zeta_j(z_j(s))]^2 \frac{\mathrm{d}}{\mathrm{d}\tau}\langle v_j^+(z)|\Phi_j(n|z)\rangle\right\}$$
$$(j = 1, 2, 3;\ s = 1, 2, 3, \ldots, (\delta_{j1} + \delta_{j3})N + \delta_{j2}M) \tag{145}$$

where under $z_2(s)$ we understand either $\overset{-}{z}_2(s)$ or $\overset{+}{z}_2(s)$ depending on which of the two equivalent expressions (115) or (116) for $|\Xi_2(n|z)\rangle$ is involved. As a result we obtain

$$\mathrm{d}z_1(s)/\mathrm{d}\tau = 0 = \mathrm{d}z_3(s)/\mathrm{d}\tau \qquad (s = 1, 2, 3, \ldots, N) \tag{146}$$

$$\mathrm{d}\overset{-}{z}_2(s)/\mathrm{d}\tau = 0 = \mathrm{d}\overset{+}{z}_2(s)/\mathrm{d}\tau \qquad (s = 1, 2, 3, \ldots, M). \tag{147}$$

After the time independence of pole locations is established, the second trick consisting of calculating

$$\lim_{n\to+\infty} \lim_{\zeta_j(z)\to\zeta_j(z_j(s))} \left\{[\zeta_j(z) - \zeta_j(z_j(s))] \frac{\mathrm{d}}{\mathrm{d}\tau}\langle v_j^+(z)|\Phi_j(n|z)\rangle\right\}$$
$$(j = 1, 2, 3;\ s = 1, 2, 3, \ldots, (\delta_{j1} + \delta_{j3})N + \delta_{j2}M) \tag{148}$$

might be done. As a result we obtain

$$\mathrm{d}C_{k1}(s)/\mathrm{d}\tau = [\eta_k(z_1(s)) - \eta_1(z_1(s))]C_{k1}(s) \qquad (k = 2, 3;\ s = 1, 2, 3, \ldots, N) \tag{149}$$

$$\mathrm{d}\overset{-}{C}_{k2}(s)/\mathrm{d}\tau = [\eta_k(\overset{-}{z}_2(s)) - \eta_2(\overset{-}{z}_2(s))]\overset{-}{C}_{k2}(s) \qquad (k = 1, 3;\ s = 1, 2, 3, \ldots, M) \tag{150}$$

$$\mathrm{d}\overset{+}{C}_{k2}(s)/\mathrm{d}\tau = [\eta_k(\overset{+}{z}_2(s)) - \eta_2(\overset{+}{z}_2(s))]\overset{+}{C}_{k2}(s) \qquad (k = 3, 1;\ s = 1, 2, 3, \ldots, M) \tag{151}$$

$$dC_{k3}(s)/d\tau = [\eta_k(z_3(s)) - \eta_3(z_3(s))]C_{k3}(s) \qquad (k = 1, 2;\ s = 1, 2, 3, \ldots, N). \tag{152}$$

The evolution equations for the complex conjugate part of scattering data can be obtained either by the same procedures with the replacement $z_j(s)$ into $z_j^*(s)$ or by the sheer complex conjugation of equations (146), (147) and (149)–(152).

11. Reconstruction of field amplitudes: general scheme

Inspecting the reconstructions (114)–(117) of envelope Jost functions $|\Xi_j(n|z)\rangle (j = 1, 2, 3)$ we observe that they can be expanded in Laurent-type series in the regions of their regularity. In particular

$$|\Xi_1(n|z)\rangle = |u_1(z)\rangle + \sum_{m=1}^{\infty} |U_1(n|m)\rangle[\zeta_1(z)]^{-m} \qquad (z \in D(231) + D(321)) \tag{153}$$

$$|\Xi_2(n|z)\rangle = |u_2(z)\rangle + \sum_{m=1}^{\infty} |U_2(n|m)\rangle[\zeta_2(z)]^{-m} \qquad (z \in D(132) + D(312)) \tag{154}$$

$$|\Xi_3(n|z)\rangle = |u_3(z)\rangle + \sum_{m=1}^{\infty} |U_3(n|m)\rangle[\zeta_3(z)]^{-m} \qquad (z \in D(123) + D(213)). \tag{155}$$

These expansions (153)–(155) enable us to reconstruct the field amplitudes $q_-(n), \alpha(n), q_+(n)$ in terms of expansion coefficients

$$\langle k|U_j(n|1)\rangle \equiv U_{kj}(n|1) \qquad (k = 1, 3;\ j = 1, 2, 3) \tag{156}$$

or more precisely in terms of their time derivatives.

For this purpose we must invoke the equalities

$$|\Xi_j(n+1|z)\rangle\zeta_j(z) = L(n|z)|\Xi_j(n|z)\rangle \qquad (j = 1, 2, 3) \tag{157}$$

for the back gauge transformed envelope Jost functions $|\Xi_j(n|z)\rangle$ on the one hand, and the evolution equations for $p_{jk}(n)$ written in their authentic form

$$\dot{p}_{jk}(n) = X_{jk}(n+1) - X_{jk}(n) \qquad (j = 1, 3;\ k = 1, 3) \tag{158}$$

on the other. Here the notations

$$X_{11}(n) = \exp[+q_-(n) - q_-(n-1)]\cos\alpha(n)\cos\alpha(n-1) \tag{159}$$

$$X_{13}(n) = \exp[+q_-(n) - q_+(n-1)]\cos\alpha(n)\sin\alpha(n-1) \tag{160}$$

$$X_{31}(n) = \exp[+q_+(n) - q_-(n-1)]\sin\alpha(n)\cos\alpha(n-1) \tag{161}$$

$$X_{33}(n) = \exp[+q_+(n) - q_+(n-1)]\sin\alpha(n)\sin\alpha(n-1) \tag{162}$$

are implied.

Collecting the lowest terms in equalities (157) expanded in accordance with the formulae (153)–(155) we might come to the interim result

$$\begin{aligned} p_{11}(n) = &-i[U_{11}(n+1|1) - U_{11}(n|1)]\exp[-q_-(-\infty)]\cos\alpha(-\infty) \\ &- i[U_{12}(n+1|1) - U_{12}(n|1)]\exp[-q_-(-\infty)]\sin\alpha(-\infty) \end{aligned} \tag{163}$$

$$\begin{aligned} p_{13}(n) = &-i[U_{13}(n+1|1) - U_{13}(n|1)]\exp[-q_+(-\infty)]\sin\alpha(-\infty) \\ &+ i[U_{12}(n+1|1) - U_{12}(n|1)]\exp[-q_+(-\infty)]\cos\alpha(-\infty) \end{aligned} \tag{164}$$

$$p_{31}(n) = -\mathrm{i}[U_{31}(n+1|1) - U_{31}(n|1)]\exp[-q_-(-\infty)]\cos\alpha(-\infty) - \mathrm{i}[U_{32}(n+1|1) - U_{32}(n|1)]\exp[-q_-(-\infty)]\sin\alpha(-\infty) \quad (165)$$

$$p_{33}(n) = -\mathrm{i}[U_{33}(n+1|1) - U_{33}(n|1)]\exp[-q_+(-\infty)]\sin\alpha(-\infty) + \mathrm{i}[U_{32}(n+1|1) - U_{32}(n|1)]\exp[-q_+(-\infty)]\cos\alpha(-\infty) \quad (166)$$

which should be inserted into the evolution equations (158) for $p_{jk}(n)$. After an appropriate summation the result for the quantities $X_{jk}(n)$ $(j = 1, 3; k = 1, 3)$ looks as follows:

$$X_{11}(n) = \cos^2\alpha(-\infty) - \mathrm{i}[\dot{U}_{11}(n|1)\cos\alpha(-\infty) + \dot{U}_{12}(n|1)\sin\alpha(-\infty)]\exp[-q_-(-\infty)] \quad (167)$$

$$X_{13}(n) = \exp[+q_-(-\infty) - q_+(-\infty)]\cos\alpha(-\infty)\sin\alpha(-\infty) - \mathrm{i}[\dot{U}_{13}(n|1)\sin\alpha(-\infty) - \dot{U}_{12}(n|1)\cos\alpha(-\infty)]\exp[-q_+(-\infty)] \quad (168)$$

$$X_{31}(n) = \exp[+q_+(-\infty) - q_-(-\infty)]\sin\alpha(-\infty)\cos\alpha(-\infty) - \mathrm{i}[\dot{U}_{31}(n|1)\cos\alpha(-\infty) + \dot{U}_{32}(n|1)\sin\alpha(-\infty)]\exp[-q_-(-\infty)] \quad (169)$$

$$X_{33}(n) = \sin^2\alpha(-\infty) - \mathrm{i}[\dot{U}_{33}(n|1)\sin\alpha(-\infty) - \dot{U}_{32}(n|1)\cos\alpha(-\infty)]\exp[-q_+(-\infty)]. \quad (170)$$

Here we should remember that neither the authentic evolution equations (158) nor the combinations of field variables (159)–(162) are independent. In order to overcome this obstacle when tackling the nonlinear dynamical system we were forced to introduce the three field parametrization (7)–(10); see also appendix B for more details. However, we do not know yet whether some universal parametrization for the expansion coefficients (156) or alternatively for the scattering data themselves is possible. For this reason we restrict ourselves to the mere statement that the property

$$X_{11}(n)X_{33}(n) = X_{13}(n)X_{31}(n) \quad (171)$$

(evident from the definitions (159)–(162)) in combination with the properties

$$U_{k1}(n|m) = U_{k3}(n|m) \qquad (k = 1, 3; m = 1, 2, 3, \ldots, \infty) \quad (172)$$

(evident from the symmetry conditions (107) and (109)) give rise to the following requirement

$$\mathrm{i}\dot{U}_{12}(n|1)\exp[+q_+(-\infty)]\sin\alpha(-\infty) - \dot{U}_{13}(n|1)\dot{U}_{32}(n|1) = \mathrm{i}\dot{U}_{32}(n|1)\exp[+q_-(-\infty)]\cos\alpha(-\infty) - \dot{U}_{31}(n|1)\dot{U}_{12}(n|1). \quad (173)$$

Inverting the definitions (159)–(162) and supposing the quantities $X_{jk}(n)$ to be given by formulae (167)–(170) under all mentioned restrictions (172) and (173), we come to the formal reconstruction of field amplitudes

$$\exp[+2q_+(n) - 2q_+(n-1)] = \frac{[X_{13}(n+1)X_{33}(n) + X_{11}(n+1)X_{13}(n)][X_{33}(n)X_{31}(n-1) + X_{31}(n)X_{11}(n-1)]}{X_{13}(n+1)X_{31}(n-1)} \quad (174)$$

$$\exp[+2q_-(n) - 2q_-(n-1)] = \frac{[X_{31}(n+1)X_{11}(n) + X_{33}(n+1)X_{31}(n)][X_{11}(n)X_{13}(n-1) + X_{13}(n)X_{33}(n-1)]}{X_{31}(n+1)X_{13}(n-1)} \quad (175)$$

$$\cos 2\alpha(n) = \frac{X_{11}(n+1)X_{13}(n) - X_{13}(n+1)X_{33}(n)}{X_{11}(n+1)X_{13}(n) + X_{13}(n+1)X_{33}(n)}. \quad (176)$$

Although the results of this section have been derived by explicitly addressing the reconstruction formulae (114)–(117) valid only for the reflectionless potentials $q_+(n)$, $\alpha(n)$, $q_-(n)$, $\dot{q}_+(n)$, $\dot{\alpha}(n)$, $\dot{q}_-(n)$, in reality however they are the most general ones inasmuch as the basic demand necessary to obtain them consists in the presentability of back-transformed Jost functions in the form of Laurent-type expansions (153)–(155). Fortunately, such expansions can be justified irrespective of any pre-assumed properties of potentials $q_+(n)$, $\alpha(n)$, $q_-(n)$, $\dot{q}_+(n)$, $\dot{\alpha}(n)$, $\dot{q}_-(n)$, namely they represent nothing but the general analytical properties of envelope Jost functions in the regions of their regularity.

12. Simplest solution

Although the full reconstruction of envelope Jost functions in the reflectionless case could, in principle, be done according to the scheme described in section 9, an extra interplay between their expansion coefficients (173) should always be taken into account or at least verified by the final results. We believe such an interplay may be linked with some intrinsic symmetries of envelope Jost functions, however at present any extra symmetries except for equations (107)–(110) have not yet been revealed and the very question about their existence remains open.

Nevertheless, bearing in mind the formal criteria already adopted for the envelope Jost functions and requiring the solutions of the nonlinear model to be finite, the correct concretizations concerning the scattering data sometimes can be made.

As an example let us analyse the simplest constructive case $N = 1$, $M = 0$, when the partial reconstructions (114)–(117) for the envelope Jost functions are taken to be

$$\begin{aligned} |\Xi_1(n|z)\rangle = |u_1(z)\rangle + |\Xi_3(n|z_1(1))\rangle C_{31}(n|1)\frac{\zeta_1(z_1(1))}{\zeta_1(z) - \zeta_1(z_1(1))} \\ + |\Xi_3(n|z_1^*(1))\rangle C_{31}^*(n|1)\frac{\zeta_1(z_1^*(1))}{\zeta_1(z) - \zeta_1(z_1^*(1))} \end{aligned} \tag{177}$$

$$|\Xi_2(n|z)\rangle = |u_2(z)\rangle. \tag{178}$$

The expression for $|\Xi_3(n|z)\rangle$ we have dropped, since it carries the same information as that already written (equation (177)).

The reconstruction of $|\Xi_1(n|z)\rangle$ will be complete provided the quantities $|\Xi_3(n|z_1(1))\rangle$ and $|\Xi_3(n|z_1^*(1))\rangle$ are found. The set of equations to find these is evidently as follows:

$$\begin{aligned} +|\Xi_3(n|z_1(1))\rangle\left[1 - C_{31}(n|1)\frac{\zeta_1(z_1(1))}{\zeta_1(z_3(1)) - \zeta_1(z_1(1))}\right] \\ - |\Xi_3(n|z_1^*(1))\rangle C_{31}^*(n|1)\frac{\zeta_1(z_1^*(1))}{\zeta_1(z_3(1)) - \zeta_1(z_1^*(1))} = |u_3(z_1(1))\rangle \end{aligned} \tag{179}$$

$$\begin{aligned} -|\Xi_3(n|z_1(1))\rangle C_{31}(n|1)\frac{\zeta_1(z_1(1))}{\zeta_1(z_3^*(1)) - \zeta_1(z_1(1))} \\ + |\Xi_3(n|z_1^*(1))\rangle\left[1 - C_{31}^*(n|1)\frac{\zeta_1(z_1^*(1))}{\zeta_1(z_3^*(1)) - \zeta_1(z_1^*(1))}\right] = |u_3(z_1^*(1))\rangle. \end{aligned} \tag{180}$$

To proceed further, the parametrization

$$z_1(1) = \exp[-\mu - \mathrm{i}k] \tag{181}$$

$$z_3(1) = \exp[+\mu + \mathrm{i}k] \tag{182}$$

by real parameters $\mu > 0$ and k seems to be reasonable. Then $C_{31}(n|1)$ can be written in the form

$$C_{31}(n|1) = \exp[+2\mu n + 2\mathrm{i}kn]C_{31}(1) \tag{183}$$

with

$$C_{31}(1) = \exp[-2\tau\,\mathrm{sh}(\mu + \mathrm{i}k) - 2\mu x(0) + \ln(2\mathrm{sh}\mu) - 2\mathrm{i}\beta(0)] \tag{184}$$

where the parameters $x(0)$ and $\beta(0)$ are real.

In order for the functions $q_-(n)$, $\alpha(n)$, $q_+(n)$ to be finite, the Cramer determinant of the linear set (179) and (180) must be of constant sign. This is achievable under restrictions $k = \pi\nu$, $\cos 2\beta(0) < 0$, where $\nu = 1, 2$ and $\beta(0) = \pi/3$ without the loss of generality.

The rest of the calculations are straightforward. The result is as follows

$$q_-(n) = \ln \frac{\mathrm{ch}[\mu(n + 1/2 - x(\tau))]}{\mathrm{ch}[\mu(n - 1/2 - x(\tau)]} + \mu + q_-(-\infty) \tag{185}$$

$$\alpha(n) = \alpha(-\infty) \tag{186}$$

$$q_+(n) = \ln \frac{\mathrm{ch}[\mu(n + 1/2 - x(\tau))]}{\mathrm{ch}[\mu(n - 1/2 - x(\tau))]} + \mu + q_+(-\infty) \tag{187}$$

where

$$x(\tau) = \tau \frac{\mathrm{sh}\mu}{\mu} \cos(\pi\nu) + x(0) \qquad (\nu = 1, 2). \tag{188}$$

13. Conclusion

Summarizing, we have developed the nonlinear model describing three coupled dynamical subsystems on a regular one-dimensional lattice. Despite highly nontrivial coupling in its kinetic and potential parts, the system as a whole admits the standard Lagrange formulation and can also be readily rewritten in the Hamiltonian form. Moreover, it was obtained as a compatibility condition of two auxiliary linear matrix equations and hence can be integrated by the method of inverse scattering transform. This method turns out to be rather complicated even in the framework of the advanced Caudrey approach, both due to the third order of the respective spectral problem and because of hidden symmetries of Jost functions yet to be revealed explicitly. We have tried to give as much information about the model as possible at present in order to make it more reliable for further investigation. It is interesting to note that even the simple result on the one-soliton solution has enabled us to test the whole inverse scattering machinery and to conclude that it should be built up on back gauge transformed envelope Jost functions $|\Xi_j(n|z)\rangle$ rather than on originally introduced ones $|\Phi_j(n|z)\rangle$.

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Appendix A. 2 × 2 matrix reduction of original discrete spectral problem

To find the relationship between the spectral problem under study (2) and the discretized Schrödinger spectral problem, it is sufficient to exclude the middle component from the original equation (2) bearing in mind the explicit expression (5) for the spectral operator $L(n|z)$. The result is as follows

$$\begin{aligned}\begin{pmatrix}\langle 1|u(n+1|z)\rangle\\ \langle 3|u(n+1|z)\rangle\end{pmatrix} - \begin{pmatrix}p_{11}(n) & p_{13}(n)\\ p_{31}(n) & p_{33}(n)\end{pmatrix}\begin{pmatrix}\langle 1|u(n|z)\rangle\\ \langle 3|u(n|z)\rangle\end{pmatrix}&\\ - \begin{pmatrix}F_{12}(n)G_{21}(n-1) & F_{12}(n)G_{23}(n-1)\\ F_{32}(n)G_{21}(n-1) & F_{32}(n)G_{23}(n-1)\end{pmatrix}\begin{pmatrix}\langle 1|u(n-1|z)\rangle\\ \langle 3|u(n-1|z)\rangle\end{pmatrix}&\\ = \lambda(z)\begin{pmatrix}\langle 1|u(n|z)\rangle\\ \langle 3|u(n|z)\rangle\end{pmatrix}&\end{aligned} \tag{A.1}$$

which is proven to be the 2 × 2 matrix version of the discrete Schrödinger spectral problem with 2 × 2 matrix potentials.

Inspecting the reduced problem (A.1) at either of the limits $n \to -\infty$ or $n \to +\infty$ we can readily conclude that it has little in common with the 2 × 2 matrix version of the discrete spectral problem considered by Bruschi and Ragnisco [50]. As for the problem (A.1) and the third-order discrete spectral problem discussed by Levi and Grundland [49], these also turn out to be different.

Appendix B. On-site conservation laws and the parametrization of spectral operator $L(n|z)$

Inserting the ansatzes (5) and (6) for the spectral $L(n|z)$ and evolution $A(n|z)$ matrices into the zero-curvature equation (1) and collecting the terms with the same powers in $\lambda(z)$ we find

$$A_{12}(n) = -F_{12}(n) \tag{B.1}$$

$$A_{21}(n) = -G_{21}(n-1) \tag{B.2}$$

$$A_{23}(n) = -G_{23}(n-1) \tag{B.3}$$

$$A_{32}(n) = -F_{32}(n) \tag{B.4}$$

and obtain the equations

$$\dot{p}_{11}(n) = F_{12}(n)G_{21}(n-1) - F_{12}(n+1)G_{21}(n) \tag{B.5}$$

$$\dot{p}_{13}(n) = F_{12}(n)G_{23}(n-1) - F_{12}(n+1)G_{23}(n) \tag{B.6}$$

$$\dot{p}_{31}(n) = F_{32}(n)G_{21}(n-1) - F_{32}(n+1)G_{21}(n) \tag{B.7}$$

$$\dot{p}_{33}(n) = F_{32}(n)G_{23}(n-1) - F_{32}(n+1)G_{23}(n) \tag{B.8}$$

$$\dot{F}_{12}(n) = p_{11}(n)F_{12}(n) + p_{13}(n)F_{32}(n) \tag{B.9}$$

$$\dot{G}_{21}(n) = -G_{21}(n)p_{11}(n) - G_{23}(n)p_{31}(n) \tag{B.10}$$

$$\dot{G}_{23}(n) = -G_{21}(n)p_{13}(n) - G_{23}(n)p_{33}(n) \tag{B.11}$$

$$\dot{F}_{32}(n) = p_{31}(n)F_{12}(n) + p_{33}(n)F_{32}(n). \tag{B.12}$$

The equations (B.5)–(B.12) are not independent although they contain all information for the nonlinear dynamical model to be extracted. Indeed we can readily verify that the two following on-site conservation laws take place:

$$F_{12}(n)G_{21}(n) + F_{32}(n)G_{23}(n) = -\exp[+2q(n)] \tag{B.13}$$

$$p_{13}(n)F_{32}(n)G_{21}(n) + p_{31}(n)F_{12}(n)G_{23}(n) - p_{11}(n)F_{32}(n)G_{23}(n) - p_{33}(n)F_{12}(n)G_{21}(n) = p(n)\exp[+2q(n)]. \tag{B.14}$$

Here $q(n)$ and $p(n)$ denote some functions of spatial coordinate n each being independent of time τ.

The first restriction (B.13) is converted into identity by the parametrization

$$F_{12}(n) = \mathrm{i}\exp[+q_-(n) + q(n) + p(n)\tau]\cos\alpha(n) \tag{B.15}$$

$$G_{21}(n) = \mathrm{i}\exp[-q_-(n) + q(n) - p(n)\tau]\cos\alpha(n) \tag{B.16}$$

$$G_{23}(n) = \mathrm{i}\exp[-q_+(n) + q(n) - p(n)\tau]\sin\alpha(n) \tag{B.17}$$

$$F_{32}(n) = \mathrm{i}\exp[+q_+(n) + q(n) + p(n)\tau]\sin\alpha(n) \tag{B.18}$$

with the quantities $q_-(n), \alpha(n), q_+(n)$ arising as three independent field variables. Then, equations (B.9)–(B.12) and the second restriction (B.14) with the use of expressions (B.15)–(B.18) yield

$$p_{11}(n) = \dot{q}_-(n) - \dot{q}_-(n)\sin^4\alpha(n) - \dot{q}_+(n)\sin^2\alpha(n)\cos^2\alpha(n) + p(n) \tag{B.19}$$

$$p_{13}(n)\exp[+q_+(n) - q_-(n)] = \dot{q}_-(n)\sin^3\alpha(n)\cos\alpha(n) + \dot{q}_+(n)\sin\alpha(n)\cos^3\alpha(n) - \dot{\alpha}(n) \tag{B.20}$$

$$p_{31}(n)\exp[-q_+(n) + q_-(n)] = \dot{q}_-(n)\sin^3\alpha(n)\cos\alpha(n) + \dot{q}_+(n)\sin\alpha(n)\cos^3\alpha(n) + \dot{\alpha}(n) \tag{B.21}$$

$$p_{33}(n) = \dot{q}_+(n) - \dot{q}_+(n)\cos^4\alpha(n) - \dot{q}_-(n)\sin^2\alpha(n)\cos^2\alpha(n) + p(n). \tag{B.22}$$

The obtained parametrization formulae (B.15)–(B.22) have to support the eigenvalues both of limiting spectral matrices $L(-\infty|z)$ and $L(+\infty|z)$ to be the same. In the case of velocities $\dot{q}_+(n), \dot{\alpha}(n), \dot{q}_-(n)$ vanishing at both spatial infinities the above demand is achievable provided the limits of $q(n)$ taken at both spatial infinities are the same

$$q(-\infty) = q = q(+\infty) \tag{B.23}$$

and the limits of $p(n)$ taken at both spatial infinities are also the same

$$p(-\infty) = p = p(+\infty). \tag{B.24}$$

Thus eight dependent variables originated from the expression (5) for the spectral operator are actually reduced to three independent field variables and their velocities.

In the main text, we have assumed $q(n)$ and $p(n)$ to be constants, which without the loss of generality are taken as zeros.

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